ON FREE OSCILLATIONS OF A VISCOUS FLUID IN A VESSEL

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A number of papers is devoted to problems of small oscillations of viscous fluids. Waves on the surface of a viscous fluid of infinite depth were examined for example in [1]. In [3] a boundary layer method is developed which is applied to problems of oscillations of a fluid in vessels in the case of small viscosity, [3-6] and others use this method to solve a series of problems on oscillations of a low viscosity fluid in certain regions. Certain general theorems on properties of characteristic oscillations of a heavy viscous fluid in a vessel are established in [7]. In [8] the approximate expression for the decrement of damping of free oscillations of a heavy viscous fluid in a cylindrical vessel of infinite depth is obtained. Results of experimental investigation of oscillations of a fluid in vessels are given in [9].

In this paper free small oscillations of a viscous incompressible fluid are studied in a stationary vessel of arbitrary shape in presence of gravity. In the main part of this paper the Reynolds' number is assumed to be large (viscosity small) which makes it possible, as in [2-6], to apply the boundary layer method. The investigation is carried out by a method which is analogous to the one which was used in [10] in the study of motion of a body with a cavity completely filled with viscous fluid. Asymptotic relationships are obtained for eigenvalues and eigenfunctions of the problem on free oscillations of a viscous fluid in an arbitrary vessel. Decrements of damping and corrections to eigenfrequencies due to viscosity are expressed through equations which depend only on the corresponding eigenfrequencies and eigenfunctions of the problem on oscillations of an ideal fluid. Computations are carried out for some specific forms of vessels.

In the last part of the paper the special character of motion of a viscous fluid near the line of contact of the free surface with the wall of the vessel is elucidated. Here free oscillations are examined for arbitrary Reynolds' number.

1. We shall examine the motion of a viscous incompressible fluid of density ρ and kinematic viscosity ν in a stationary vessel (Fig. 1). The equations of motion of the fluid have the form

$$\mathbf{U}_t + (\mathbf{U}\nabla)\mathbf{U} = -\rho^{-1}\nabla P - g\mathbf{k} + \nu\Delta\mathbf{U}, \qquad \operatorname{div}\mathbf{U} = 0 \tag{1.1}$$

Here t is the time, $\mathbf{U}$ is the velocity of the fluid, P is the pressure, g is the acceleration due to gravity, unit vector $\mathbf{k}$ is directed vertically upward, and index t indicates a partial derivative. Assume that L is a characteristic linear dimension of the vessel, $T = (L/g)^{1/2}$ is the characteristic time (of the order of period of oscillations), l is the characteristic amplitude of oscillation of fluid particles and the Reynolds' number R is large

$$R = L^2 T^{-1} \nu^{-1} = L^{3/2} g^{1/2} \nu^{-1} \gg 1 \tag{1.2}$$

Equation (1.1) can be linearized if $|(\mathbf{U} \nabla)\mathbf{U}| \ll |\mathbf{U}_t|$. With respect to the order of magnitude we have: $|\mathbf{U}| \sim l\,T^{-1}$, $|\mathbf{U}_t| \sim l\,T^{-2}$. For operator ∇ outside the boundary layer the estimate $\nabla \sim L^{-1}$, is correct, therefore $|\mathbf{U}\nabla| \sim lL^{-1}T^{-1}$. In the boundary layer individual components of vectors $\mathbf{U}$ and ∇ have different orders of magnitude (see below), however, the order of magnitude $|\mathbf{U}\nabla|$ is here the same as outside the boundary layer. Therefore the condition of linearization is reduced to the form

$$l \ll L \tag{1.3}$$

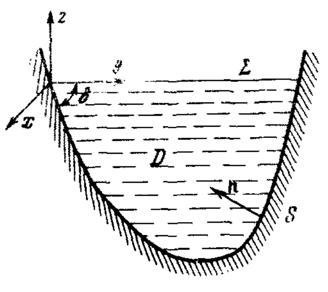

FIG. 1

and in the following it will be assumed to hold. Below the problem is examined in a linear formulation and is solved in the form of series in terms of parameter $R^{-1/2} \ll 1$. Since it is desirable that the error due to non-linearity which is $O\,(l/T)$, does not exceed terms of the n-th approximation in the solution of the linear problem, we must impose the condition $l\,/\,L \ll R^{-n/2}$, which is stronger than (1.3) (see also [5]). For computation of the decrement of damping in the first approximation the fulfillment of condition (1.3) is sufficient. We can write the linearized equations of motion and boundary conditions, in the form

$$\mathbf{U}_t = -\rho^{-1}\nabla P - g\mathbf{k} + \nu\Delta\mathbf{U}, \qquad \operatorname{div}\mathbf{U} = 0 \text{ in } D, \qquad \mathbf{U} = 0 \text{ on } S$$

$$P + \frac{\partial P}{\partial z}F - 2\rho\nu\frac{\partial U_z}{\partial z} = P_0, \qquad \frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} = \frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} = 0 \tag{1.4}$$

$$U_z = \frac{\partial F\,(x, y, t)}{\partial t} \quad \text{on } \Sigma$$

Here D is the region occupied by the fluid at rest and S and Σ are the wetted wall surface and the free surface, respectively, in the condition of rest. A Cartesian system of coordinates xyz is selected such that the plane xy coincides with the unperturbed free surface Σ, while the z-axis is oriented vertically upward (Fig. 1). Conditions on free surface expressing the equality of forces inside and outside the fluid (P_0 is the constant pressure outside the fluid) are taken with respect to Σ, while $z = F\,(x, y, t)$ is the equation of the perturbed free surface.

We are looking for the solution of the problem of free oscillations of a fluid in the form

$$\mathbf{U} = e^{\lambda t}\mathbf{u}, \qquad P = P_0 - \rho g z + \rho e^{\lambda t} q, \qquad F = e^{\lambda t} f\,(x, y) \tag{1.5}$$

where λ is a complex eigenvalue and $\mathbf{u}$ and q are functions of coordinates x, y and z. Substituting (1.5) into (1.4) we arrive at the eigenvalue problem

$$\lambda \mathbf{u} = -\nabla q + \nu \Delta \mathbf{u}, \qquad \operatorname{div} \mathbf{u} = 0 \ \text{ in } D, \qquad \mathbf{u} = 0 \ \text{ on } S$$
$$u_z = \frac{\lambda}{g} q - \frac{2\lambda\nu}{g} \frac{\partial u_z}{\partial z}, \qquad \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0, \ f = \frac{u_z}{\lambda} \ \text{ on } \Sigma \tag{1.6}$$

It is required to determine values $\lambda \neq 0$ for which the boundary value problem (1.6) admits a nonzero solution, and to find eigenfunctions $\mathbf{u}$ and q for these λ. The last condition (1.6) can be used for determination of the form of the free surface after solution of the problem. Through appropriate choice ofunits of length and time measurement it is possible to achieve that $L \sim 1$ and $g \sim 1$. Then, by virtue of (1.2), problem (1.6) will contain a small parameter $\nu \ll 1$.

2. The solution of problem (1.6) is sought, as in [2 to 6 and 10], by the boundary layer method [11]. We assume

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \qquad q = s + h, \qquad \mathbf{v} = \mathbf{v}^\circ + \nu^{1/2}\mathbf{v}^1 + \ldots$$
$$s = s^\circ + \nu^{1/2}s^1 + \ldots, \ \lambda = \lambda^\circ + \nu^{1/2}\lambda^1 + \ldots \tag{2.1}$$

Here $\mathbf{w}$ and h are functions of the type of boundary layer. These functions can also be expanded in series of powers of $\nu^{1/2}$, where all coefficients of their expansions $\mathbf{w}^k$, h^k diminish rapidly with increase of distance from the boundaries of region D. Let us designate by D_S and D_Σ regions of boundary layer adjacent from the inside to surfaces S and Σ, respectively, and of thickness of the order of $\nu^{1/2}$. Then it can be assumed that $\mathbf{w} = 0$ and $h = 0$ outside D_S and D_Σ.

Let us require that the functions $\mathbf{v}$ and s and also $\mathbf{w}$ and h, satisfy Equations (1.6). Boundary conditions for these can be obtained in the following manner. Let functions $\mathbf{v}^i$, s^i, $\mathbf{w}^i$, h^i, λ^i, where $i = 0, 1 \ldots, k-1$, be already determined. For determination of $\mathbf{v}^k$ and s^k it will be required that these functions, together with $\mathbf{v}^i$, s^i, $\mathbf{w}^i$, and h^i found earlier, satisfy the condition $\mathbf{un} = 0$ at the wall S and the first of conditions of (1.6) on Σ. Here $\mathbf{n}$ is the unit vector of the inward normal to S.

In the solution of the boundary value problem for $\mathbf{v}^k$ and s^k the value λ^k will also be determined at the same time. Then we shall determine the functions $\mathbf{w}^k$ and h^k which, taking into account $\mathbf{v}^i$, s^i, $\mathbf{w}^i$, h^i, λ^i, $\mathbf{v}^k$, s^k, and λ^k already found, must satisfy the condition of vanishing of components of vector $\mathbf{u}$ tangential to wall S, two conditions of (1.6) of vanishing of tangential forces on Σ, and also the conditions that $\mathbf{w}^k \to 0$ and $h_k \to 0$ outside D_S and D_Σ. It is not difficult to convince oneself that the described process for constructing the solution fomally ensures satisfaction of equations and boundary conditions with an error of the order of $\nu^{(k+1)/2}$, which tends to zero with increase of the numbers of approximation k.

Since functions $\mathbf{v}$ and s represented by series (2.1) satisfy Equations (1.6), we find for the k-th approximation

$$\sum_{i=0}^{k} \lambda^i \mathbf{v}^{k-i} = -\nabla s^k + \Delta \mathbf{v}^{k-2}, \qquad \operatorname{div} \mathbf{v}^k = 0 \qquad (k = 0, 1, \ldots) \tag{2.2}$$

where the term $\Delta \mathbf{v}^{k-2}$ in the first equation of (2.2) is not present when $k = 0, 1$. If all $\lambda^i \neq 0$, then from (2.2) it is easy to obtain by induction that all $\mathbf{v}^k$ are potential vectors in D and $\Delta \mathbf{v}^k = 0$. Then, by virtue of (2.2) we have

$$\mathbf{v}^k = \nabla\varphi^k,\ \Delta\varphi^k = 0, \qquad s^k = -\sum_{i=0}^{k} \lambda^i\varphi^{k-i} \qquad (k = 0, 1, \ldots) \tag{2.3}$$

In the last of equations of (2.3) the fact that function φ^k is determined with accuracy to an arbitrary function of time, is already utilised. Below we limit ourselves to the examination of $\mathbf{v}^\circ$, s°, λ°, $\mathbf{v}^1$, s^1, λ^1, and also $\mathbf{w}^\circ$ and h° which will simply be denoted by $\mathbf{w}$, h.

As follows from the above-described process of determining the solution, we have for functions $\mathbf{v}^\circ$ and s° the boundary conditions $\mathbf{v}^\circ\mathbf{n} = 0$ and $\lambda^\circ s^\circ = g v_z{}^\circ$ on Σ.

Utilizing Equations (2.3), we obtain

$$\Delta\varphi^\circ = 0 \ \text{ in } D, \qquad \frac{\partial\varphi^\circ}{\partial n} = 0 \ \text{ on } S, \qquad \frac{\partial\varphi^\circ}{\partial z} + \frac{(\lambda^\circ)^2}{g}\varphi^\circ = 0 \ \text{ on } \Sigma$$
$$\mathbf{v}^\circ = \nabla\varphi^\circ, \qquad s^\circ = -\lambda^\circ\varphi^\circ \tag{2.4}$$

The eigenvalue problem (2.4) for φ° describes free oscillations of an ideal fluid and has a discrete spectrum of purely imaginary eigenvalues $\lambda^\circ = \pm i\omega_m$, $\omega_m > 0$, $m = 1, 2, \ldots$, with finite mutliplicity [12]. The eigenfunction Φ_m corresponding to the frequency ω_m satisfies the boundary value problem

$$\Delta\Phi_m = 0 \ \text{ in } D,\ \partial\Phi_m / \partial n = 0 \ \text{ on } S,\ \partial\Phi_m / \partial z = (\omega_m{}^2 / g)\ \Phi_m \text{ on } \Sigma \tag{2.5}$$

We shall look for characteristic oscillation of a viscous fluid which, as $\nu \to 0$, passes into the m-th oscillation of an ideal fluid, i.e. we shall write

$$\varphi^\bullet = \Phi_m, \qquad \lambda^\circ = \pm i\omega_m$$

For functions $\mathbf{w}$ and h we have, in accordance with what was said above, the boundary value problem

$$\lambda^\circ\mathbf{w} = -\nabla h + \nu\Delta\mathbf{w}, \qquad \operatorname{div}\mathbf{w} = 0 \quad \text{in } D_S,\ D_\Sigma$$
$$\mathbf{w}^* = -\mathbf{v}^\circ \ \text{ on } S, \qquad \mathbf{w} \to 0,\ h \to 0 \ \text{ outside } D_S,\ D_\Sigma \tag{2.6}$$
$$\frac{\partial(w_x + v_x{}^\circ)}{\partial z} + \frac{\partial(w_z + v_z{}^\circ)}{\partial x} = \frac{\partial(w_y + v_y{}^\circ)}{\partial z} + \frac{\partial(w_z + v_z{}^\circ)}{\partial y} = 0 \ \text{ on } \Sigma$$

Here $\mathbf{w}^*$ is the projection of vector $\mathbf{w}$ on the plane tangential to S; vector $\mathbf{v}^\circ$ on S also lies in this plane by virtue of relationship $\mathbf{v}^\circ\mathbf{n} = 0$. Let us find the asymptotic solution of the problem (2.6) separately in regions D_S and D_Σ as $\nu \to 0$. In the region D_Γ, which is the intersection of these two regions and which is adjacent to the contour Γ (the line of contact of free surface Σ and walls of vessel S) the solution has a more complicated character.

In the region D_S we introduce curvilinear orthogonal coordinates $\xi\eta\zeta$ such that the surface $\zeta = 0$ coincides with surface S and such, that in the region D_S, $\zeta > 0$. Let H_ξ, H_η, and H_ζ denote the corresponding Lamé's constants, $H_\xi{}^\circ$, $H_\eta{}^\circ$, and $H_\zeta{}^\circ$ their values when $\zeta = 0$ and w_ξ, w_η, and w_ζ the components of vector $\mathbf{w}$ in these coordinates. Without loosing generality we set $H_\zeta{}^\circ = 1$. Then, with accuracy to the infinitesimals of higher order, ζ is the distance from surface S along the inner normal and w_ζ is the projection of **w** on the normal **n**. We also write

$$\zeta = \nu^{1/2}\alpha, \qquad w_\zeta = \nu^{1/2} w_\alpha \tag{2.7}$$

and pass in Equations (2.6) to variables ξ, η and α. Taking into account that $\zeta \sim \nu^{1/2}$, and $\alpha \sim 1$ in D_S and omitting infinitesimals of the order $\nu^{1/2}$ and higher in equations of motion (2.6), we reduce these equations to the following form (see also [10])

$$\frac{\partial h}{\partial \alpha} = 0, \qquad \lambda^\circ w_\xi = -\frac{\partial h}{\partial \xi} + \frac{\partial^2 w_\xi}{\partial \alpha^2}, \qquad \lambda^\circ w_\eta = -\frac{\partial h}{\partial \eta} + \frac{\partial^2 w_\eta}{\partial \alpha^2}$$
$$\frac{\partial}{\partial \xi}(H_\eta{}^\circ w_\xi) + \frac{\partial}{\partial \eta}(H_\xi{}^\circ w_\eta) + H_\xi{}^\circ H_\eta{}^\circ \frac{\partial w_\alpha}{\partial \alpha} = 0 \tag{2.8}$$

Since $h \to 0$ outside region D_S, i.e. when $\alpha \to \infty$, it follows from (2.8) that $h \equiv 0$. Then we obtain the following boundary value problem for **w**, taking into account condition (2.6) on S

$$\lambda^\circ \mathbf{w}^* = \frac{\partial^2 \mathbf{w}^*}{\partial \alpha^2}, \qquad \mathrm{Div}\, \mathbf{w}^* + \frac{\partial w_\alpha}{\partial \alpha} = 0$$
$$\mathbf{w}^* = -\mathbf{v}^\circ \text{ when } \alpha = 0, \qquad \mathbf{w}^* \to 0, \quad w_\alpha \to 0 \text{ when } \alpha \to \infty \tag{2.9}$$

Here $\mathbf{w}^*$ is a vector with components w_ξ, and w_η, and Div designates divergence operation with respect to two-dimensional vector on the surface S.

From (2.9) we shall determine first $\mathbf{w}^*$ and then w_α

$$\mathbf{w}^* = -\mathbf{v}^\circ \exp(\sqrt{\lambda^\circ}\alpha), \qquad w_\alpha = (\mathrm{Div}\, \mathbf{v}^\circ / \sqrt{\lambda^\circ}) \exp(\sqrt{\lambda^\circ}\alpha)$$

where we have selected the branch $\sqrt{\lambda^\circ}$, for which Re $\sqrt{\lambda^\circ} < 0$. Returning to variables ζ and w_ζ, according to (2.7) and taking into account the equality $\mathbf{v}^\circ = \nabla \varphi^\circ$, we obtain finally in D_S

$$\mathbf{w}^*(\xi, \eta, \zeta) = -\nabla\varphi^\circ(\xi, \eta, 0)\exp(\sqrt{\lambda^\circ/\nu}\,\zeta), \qquad h \equiv 0$$
$$w_\zeta(\xi, \eta, \zeta) = (\sqrt{\nu}/\sqrt{\lambda^\circ})\,\mathrm{Div}\,[\nabla\varphi^\circ(\xi, \eta, 0)]\exp(\sqrt{\lambda^\circ/\nu}\,\zeta) \tag{2.10}$$

In region D_Σ we write by analogy to (2.7)

$$z = \nu^{1/2}\beta, \qquad w_z = \nu^{1/2} w_\beta \tag{2.11}$$

and pass in equations and boundary conditions (2.6) to variables x, y and β, neglecting the infinitesimals of higher order in ν. By analogy to (2.8) and (2.9) we obtain Equations

$$h \equiv 0, \quad \lambda^\circ w_x = \frac{\partial^2 w_x}{\partial \beta^2}, \qquad \lambda^\circ w_y = \frac{\partial^2 w_y}{\partial \beta^2}, \qquad \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_\beta}{\partial \beta} = 0 \tag{2.12}$$

Boundary conditions (2.6) on Σ and conditions on the boder of the boundary layer yield in the first approximation, with consideration of Equations (2.4) and (2.11),

$$\frac{\partial w_x}{\partial \beta} = -\nu^{1/2}\left(\frac{\partial v_x{}^\circ}{\partial z} + \frac{\partial v_z{}^\circ}{\partial x}\right) = -2\nu^{1/2}\frac{\partial^2 \varphi^\circ}{\partial x \partial z}, \qquad \frac{\partial w_y}{\partial \beta} = -2\nu^{1/2}\frac{\partial^2 \varphi^\circ}{\partial y \partial z} \quad \text{when } \beta = 0$$
$$w_x, w_y, w_\beta \to 0 \quad \text{as} \quad \beta \to -\infty \tag{2.13}$$

Let us solve Equations (2.12) with conditions (2.13) (first for w_x and w_y then for w_β) and change to the variables z and w_z

$$w_x(x,\,y,\,z) = 2\left(\frac{\nu}{\lambda^\circ}\right)^{1/2}\frac{\partial^2\varphi^\circ}{\partial x\partial z}(x,\,y,\,0)\exp\left(-\left(\frac{\lambda^\circ}{\nu}\right)^{1/2}z\right), \quad \operatorname{Re}\sqrt{\lambda^\circ} < 0$$

$$w_y(x,\,y,\,z) = 2\left(\frac{\nu}{\lambda^\circ}\right)^{1/2}\frac{\partial^2\varphi^\circ}{\partial y\partial z}(x,\,y,\,0)\exp\left(-\left(\frac{\lambda^\circ}{\nu}\right)^{1/2}z\right) \tag{2.14}$$

$$w_z(x,\,y,\,z) = -\frac{2\nu}{\lambda^\circ}\frac{\partial^3\varphi^\circ}{\partial z^3}(x,\,y,\,0)\exp\left(-\left(\frac{\lambda^\circ}{\nu}\right)^{1/2}z\right), \qquad h \equiv 0$$

Here Laplace's equation for φ° was used. Solutions of the form (2.14) were obtained in [3]. As was assumed, above solutions (2.10) and (2.14) decay rapidly (exponentially) when $\zeta \gg \nu^{1/2}$, $|z| \gg \nu^{1/2}$, i.e. outside the boundary layer. As was noted in [3], the components of vector $\mathbf{w}$ tangential and normal to the boundary of region D, have different orders of magnitude; moreover these components are larger in D_S, than in D_Σ (see (2.10) and (2.14)).

Let us estimate functions $\mathbf{w}$ and h in region D_Γ, adjacent to contour Γ. Thickness of this region along the normal to Γ is of the order of $\nu^{1/2}$, and for the differentiation operator in D_Γ, generally speaking, the estimate $|\nabla| \sim \nu^{-1/2}$ is appropriate.

Therefore, since $|\mathbf{w}| \leqslant 0\ (1)$ outside D_Γ, in D_Γ we will have $|\mathbf{w}| \sim 1$ and $\partial w_z/\partial z \sim \nu^{-1/2}$. From Equations (2.6) it then follows that $|\nabla h| \sim 1$, and, since $h = 0$ outside D_Γ, then $h \sim \nu^{1/2}$ in D_Γ. The obtained estimates

$$|\mathbf{w}| \sim 1, \ \ \partial w_z/\partial z \sim \nu^{-1/2}, \qquad h \sim \nu^{1/2} \ \text{ in } D_\Gamma \tag{2.15}$$

will be used subsequently.

Functions $\mathbf{v}^1$ and s^1 must compensate for the discrepancy in fulfillment of condition $\mathbf{un} = 0$ and S and of first condition of (1.6) on Σ. The discrepancy is due to solution in the boundary layer of $\mathbf{w}$ and h. We shall write these conditions substituting $\mathbf{u}$, q and λ into them according to Equations (2.1) and (2.3) (with accuracy to the infinitesimals of higher order)

$$\mathbf{u} = \nabla\varphi^\circ + \nu^{1/2}\nabla\varphi^1 + \mathbf{w}, \qquad q = -\lambda^\circ\varphi^\circ - \nu^{1/2}(\lambda^\circ\varphi^1 + \lambda^1\varphi^\circ) + h \tag{2.16}$$
$$\lambda = \lambda^\circ + \nu^{1/2}\lambda^1$$

and taking into account boundary conditions (2.4) for φ°

$$\frac{\partial\varphi^1}{\partial n} = -\frac{\mathbf{wn}}{\sqrt{\nu}} \ \text{ on } S, \qquad \frac{\partial\varphi^1}{\partial z} = -\frac{(\lambda^\circ)^2\varphi^1 + 2\lambda^\circ\lambda^1\varphi^\circ}{g} - $$
$$-\frac{w_z}{\sqrt{\nu}} + \frac{\lambda^\circ}{g\sqrt{\nu}}h - \frac{2\lambda^\circ\sqrt{\nu}}{g}\frac{\partial w_z}{\partial z} \ \text{ on } \Sigma \tag{2.17}$$

Here the orders of magnitude of w_z and h on Σ are taken into consideration and infinitesimals of higher order have been dropped. We note that almost in the entire region of Σ (with the exception of a narrow region with a width of the order of $\nu^{1/2}$, adjacent to contour Γ) it is also possible to discard the last three terms in the second condition of (2.17), since they will be small (no larger than $\nu^{1/2}$) by virtue of (2.14).

In the adopted approximation the solution of problem (1.6) is determined by Equations (2.16) in which $\mathbf{w}$ and h outside D_Γ are given by Equations (2.10) and (2.14). For determination of φ°, λ°, φ^1, and λ^1 it is necessary to solve problems (2.4) and (2.17) for the Laplace's equation.

The correction λ^1 to the eigenvalue is of the greatest interest. It turns out that it can be expressed in terms of λ° and φ° only. Into Green's equation for functions φ°, and φ^1 harmonic in D

$$\int_S \left(\varphi^\circ \frac{\partial \varphi^1}{\partial n} - \varphi^1 \frac{\partial \varphi^\circ}{\partial n}\right) ds + \int_\Sigma \left(\varphi^1 \frac{\partial \varphi^\circ}{\partial z} - \varphi^\circ \frac{\partial \varphi^1}{\partial z}\right) ds = 0 \qquad (2.18)$$

we substitute normal derivatives of these functions on S and Σ according to (2.4) and (2.17). We note that the last two terms in the second equation of (2.17) are finite (not small) only in the region with area $\sim \nu^{1/2}$ near contour Γ, where, according to estimates (2.15), they have the order of magnitude O (1). Therefore these terms will make a contribution of the order of $\nu^{1/2}$ to the integral (2.18) over Σ and can be dropped. The term $(-\nu^{1/2} w_z)$ cannot be neglected from (2.17) since it will make a finite contribution in the integration over Σ. After indicated transformations Equation (2.18) is brought to the form

$$-\int_S \frac{\varphi^\circ \mathbf{wn}}{\sqrt{\nu}} ds + \int_\Sigma \frac{2\lambda^\circ \lambda^1 (\varphi^\circ)^2}{g} ds + \int_\Sigma \frac{\varphi^\circ w_z}{\sqrt{\nu}} ds = 0$$

From this we find, by utilizing the theorem of Gauss-Ostrogradskii and the equation $\operatorname{div} \mathbf{w} = 0$

$$\frac{2\lambda^\circ \lambda^1}{g} \int_\Sigma (\varphi^\circ)^2 ds = -\int_D \frac{\operatorname{div}(\varphi^\circ \mathbf{w})}{\sqrt{\nu}} dV = -\int_D \frac{\nabla \varphi^\circ \cdot \mathbf{w}}{\sqrt{\nu}} dV \qquad (2.19)$$

Function $\mathbf{w}$ is finite in D_S and D_Γ, and small in D_Σ (see (2.10), (2.14) and (2.15)) and practically equal to zero in the remaining part of region D. Since region D_S has a volume $\sim \nu^{1/2}$ and D_Γ of the order ν, the main contribution to the integral (2.19) over D will be made by the integral over D_S. In the region D_S it is possible to assume $\mathbf{w} = \mathbf{w}^*$ with accuracy to infinitesimals of higher order (see (2.10)). Function $\nabla \varphi^\circ$ can be evaluated at $\zeta = 0$, i.e. on the wall S. Furthermore, since $\mathbf{w}^*$ rapidly decreases with increasing ζ in the region D_S, the integration over D_S can be reduced to integrating first over ζ from 0 to ∞ and subsequently over the surface S. Then Equation (2.19) is converted into the following form

$$\frac{2\lambda^\circ \lambda^1}{g} \int_\Sigma (\varphi^\circ)^2 ds = -\int_{D_S} \frac{\nabla \varphi^\circ \cdot \mathbf{w}^*}{\sqrt{\nu}} dV = -\int_S \nabla \varphi^\circ \left(\int_0^\infty \frac{\mathbf{w}^* d\zeta}{\sqrt{\nu}}\right) ds = -\int_S \frac{(\nabla \varphi^\circ)^2 ds}{\sqrt{\lambda^\circ}} \qquad (2.20)$$

In the integration over ζ, Equation (2.10) for $\mathbf{w}^*$ was utilized.

We shall now quote the expression for $\sqrt{\lambda^\circ}$ when $\operatorname{Re} \sqrt{\lambda^\circ} < 0$

$$\lambda^\circ = \pm i\omega_m, \qquad \sqrt{\lambda^\circ} = -[(1 \pm i)/\sqrt{2}]\sqrt{\omega_m}, \qquad \omega_m > 0$$

We substitute into Equation (2.20) expressions for λ°, $\sqrt{\lambda^\circ}$ and $\varphi^\circ = \Phi_m$. Equation (2.20) is solved for λ^1 and subsequently λ^1 is substituted into Equation (2.16) for λ.

Finally we obtain

$$\lambda_m = \pm i\omega_m - \frac{(1 \pm i)\nu^{1/2} g}{2\sqrt{2}\omega_m^{3/2}} A_m, \quad A_m = \left[\int_S (\nabla \Phi_m)^2 ds\right] \Big/ \left(\int_\Sigma \Phi_m^2 ds\right) \quad (m = 1, 2, \ldots) \qquad (2.21)$$

Here $\lambda = \lambda_m$ is the eigenvalue of the problem (1.6), close to the m-th eigenvalue of the problem on oscillations of ideal fluid.

Equation (2.21) shows that viscosity leads to the appearance of a damping decrement of characteristic oscillations (Re $\lambda_m < 0$) and to a decrease of fundamental frequency by an amount equal to this decrement. We note that the numerator of Expression (2.21) for A_m is proportional to energy dissipation in the boundary layer while the denominator is proportional to the kinetic energy of oscillation of an ideal fluid. In [7] it is pointed out that problem (1.6) has when $\nu > 0$, a finite number of complex eigenvalues. Equation (2.21) is applicable, only to a finite number of frequencies, i.e. for $m < m_0$, where $m_0 \to \infty$ as $\nu \to 0$.

Equation (2.21) is also valid in the case of plane oscillations of a fluid in an infinite cylindrical vessel (channel) the generators of which are horizontal and perpendicular to the plane of motion. In this case D must be taken as the cross-section of the cylinder by the plane which is perpendicular to the generators, S must be taken as the curve along which the plane intersects the walls of the vessel, and section Σ as the section of free surface by the same plane.

3. For computation of λ_m from Equation (2.21) it is sufficient to solve the eigenvalue problem (2.5). This problem has been solved analytically or numerically for many shapes of vessels, therefore computation of λ_m for these vessels is reduced to computation of quadratures. We shall examine some examples.

Let the vessel have vertical walls and a flat bottom. The depth of fluid is constant and equal to H. Solution of problem (2.5) in this case [1] can be sought in the form

$$\Phi_m = \psi_m\,(x, y) \cosh k_m\,(z + H) \tag{3.1}$$

Here function ψ_m is the solution of the eigenvalue problem

$$\Delta\psi_m + k_m^2\psi_m = 0 \quad \text{in} \;\; \Sigma, \qquad \partial\psi_m/\partial N = 0 \quad \text{on} \;\; \Gamma \tag{3.2}$$

where Δ is the Laplace's operator in the xy-plane and N is the normal to contour Γ lying in this plane. Frequencies ω_m are expressed through values k_m by the equation

$$\omega_m^2 = gk_m \;\tanh\,(k_m H) \tag{3.3}$$

Substituting (3.1) into Equation (2.21) for A_m, we obtain after elementary integration with respect to z

$$\begin{gathered} A_m = \left\{\frac{\operatorname{th}(k_m H)}{2k_m}\int\limits_\Gamma [(\nabla\psi_m)^2 + k_m^2\psi_m^2]\,dl \pm \frac{H}{2\cosh^2(k_m H)}\int\limits_\Gamma [(\nabla\psi_m)^2 - \right. \\ \left. - k_m^2\psi_m^2]\,dl + \frac{1}{\cosh^2(k_m H)}\int\limits_\Sigma (\nabla\psi_m)^2 ds\right\}\left(\int\limits_\Sigma \psi_m^2 ds\right)^{-1} \end{gathered} \tag{3.4}$$

Eigenvalues λ_m are determined by the general equation (2.21) in which ω_m and A_m are given by Equations (3.3) and (3.4). Considering that k_m and ψ_m do not depend on H, we simplify expressions (3.4) and (2.21) for A_m and λ_m for the cases of infinitely great and infinitely small depth of fluid

$$A_m = \frac{1}{2k_m}\left\{\int\limits_\Gamma [\nabla(\psi_m)^2 + k_m^2\psi_m^2]\,dl\right\}\left(\int\limits_\Sigma \psi_m^2 ds\right)^{-1}, \qquad \omega_m^2 = gk_m \tag{3.5}$$

$$\lambda_m = \pm i\omega_m - \frac{(1\pm i)\,\nu^{1/2}g^{1/4}}{2\sqrt{2}\,k_m^{3/4}}\,A_m \qquad \text{when} \quad H=\infty$$

$$A_m = \Big[\int\limits_{\Sigma} (\nabla\psi_m)^2 ds\Big]\Big(\int\limits_{\Sigma} \psi_m{}^2 ds\Big)^{-1}, \qquad \omega_m{}^2 = gk_m{}^2 H$$

$$\lambda_m = \pm i\omega_m - \frac{(1\pm i)\,\nu^{1/2}g^{1/4}}{2\sqrt{2}H^{3/4}k_m^{3/2}}\,A_m \qquad \text{as} \quad H\to 0$$

Let us examine a vessel in the shape of a rectangular parallelepiped for which the region Σ is a rectangle $0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b$.

Solution of problem (3.2) in this case has the form

$$\psi_{mn} = \cos\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{b}\right), \qquad k_{mn}{}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right) \qquad (m,\ n=0,\ 1,\ldots) \quad (3.6)$$

Simple computation by means of Equation (3.4) gives

$$\begin{aligned} A_{mn} = \frac{2\tanh(k_{mn}H)}{k_{mn}}\left[\frac{\pi^2 m^2}{a^2}\left(\frac{1}{a}+\frac{2-\delta_{n0}}{b}\right)+\frac{\pi^2 n^2}{b^2}\left(\frac{1}{b}+\frac{2-\delta_{m0}}{a}\right)\right] - \\ - \frac{2H}{\mathrm{ch}^2(k_{mn}H)}\left(\frac{\pi^2 m^2}{a^3}+\frac{\pi^2 n^2}{b^3}\right)+\frac{k_{mn}{}^2}{\mathrm{ch}^2(k_{mn}H)} \quad (m^2+n^2\neq 0) \end{aligned} \quad (3.7)$$

where δ_{m0} and δ_{n0} are Kronecker deltas. Equations (2.21), (3.3), (3.6) and (3.7) determine the solution of the problem in the case of a rectangular parallelepiped. Assuming that $n = 0$ and $b \gg a$ in (3.6) and (3.7), we obtain the following equation for the case of plane transverse oscillations of a fluid in a long rectangular channel

$$A_{m0} = \frac{\pi^2 m^2}{a^2}\left[\frac{2\tanh(k_{m0}H)}{\pi m}+\frac{1-(2H/a)}{\mathrm{ch}^2(k_{m0}H)}\right], \qquad k_{m0} = \frac{\pi m}{a} \qquad (m=1,\ 2,\ \ldots) \quad (3.8)$$

Here a is the width and H the depth of the channel.

If we also write $H = \infty$, Equations (3.8) and (3.5) will give characteristic frequencies of plane waves of a viscous fluid confined between two parallel vertical planes

$$\lambda_m = \pm i\left(\frac{\pi m g}{a}\right)^{1/2} - (1\pm i)\left(\frac{\pi m g\nu^2}{4a^5}\right)^{1/4} \qquad (m=1,\ 2,\ \ldots)$$

This case was examined in [4] where the same expression was obtained for λ_m. In the other limiting case of a channel of small depth ($H \ll a$) we find from (3.8) and (3.5)

$$\lambda_m = \pm i\frac{\pi m}{a}(gH)^{1/2} - (1\pm i)\left(\frac{\pi^2 m^2 g\nu^2}{64a^2H^3}\right)^{1/4} \qquad (m=1,\ 2,\ \ldots)$$

Now we shall examine a vessel in the shape of a right circular cylinder of radius a and depth H. Solution of problem (3.2) for a circle $r\leqslant a$ can be taken in the form

$$\begin{aligned} &\psi_{0n} = 2^{-1/2}J_0(\mu_{0n}r/a), \qquad \psi_{mn}{}^{(1)} = J_m(\mu_{mn}r/a)\cos m\varphi \\ &\psi_{mn}{}^{(2)} = J_m(\mu_{mn}r/a)\sin m\varphi \qquad (m,\ n=1,\ 2,\ \ldots) \end{aligned} \quad (3.9)$$

Here r and φ are polar coordinates in the xy-plane with the center on the axis of the cylinder, and μ_{mn} are consecutive positive roots of derivatives of Bessel functions

$$J_m'(\mu_{mn}) = 0, \qquad 0<\mu_{m1}<\mu_{m2}<\ldots \qquad \begin{pmatrix} m=0,\ 1,\ \ldots \\ n=1,\ 2,\ \ldots \end{pmatrix} \quad (3.10)$$

The numbers k_{mn} are connected with μ_{mn} through relationships $ak_{mn} = \mu_{mn}$, where

$m = 0, 1, \ldots, n = 1, 2, \ldots$, while the frequencies ω_{mn} are expressed through k_{mn} by means of the general formula (3.3). The eigenvalues k_{mn} for $m > 0$ are double valued.

Let us compute for Function (3.9) the integrals which enter into Equation (3.4)

$$\int_\Gamma (\nabla \psi_{mn})^2 \, dl = \frac{\pi m^2}{a} J_m{}^2(\mu_{mn}), \qquad \int_\Gamma \psi_{mn}{}^2 dl = \pi a J_m{}^2(\mu_{mn})$$

$$\int_\Sigma \psi_{mn}{}^2 ds = \frac{\pi a^2}{\mu_{mn}{}^2} \int_0^{\mu_{mn}} J_m{}^2(x)\, x dx = \frac{\pi a^2}{2\mu_{mn}{}^2} (\mu_{mn}{}^2 - m^2) J_m{}^2(\mu_{mn})$$

$$\int_\Sigma (\nabla \psi_{mn})^2 ds = \pi \int_0^{\mu_{mn}} \left[J_m'^2(x) + \frac{m^2}{x^2} J_m{}^2(x) \right] x dx = \pi \int_0^{\mu_{mn}} \left[-J_m (x J_m')' + \right.$$

$$\left. + \frac{m^2}{x} J_m{}^2 \right] dx = \pi \int_0^{\mu_{mn}} J_m{}^2(x)\, x dx = \frac{\pi}{2} (\mu_{mn}{}^2 - m^2) J_m{}^2(\mu_{mn})$$

In the transformations some equalities for Bessel functions [13] were used and also the condition (3.10). Substituting results of computations into Equations (3.4) and (2.21) we find

$$A_{mn} = \frac{\mu_{mn}{}^2}{a^2} \left[\frac{\mu_{mn}{}^2 + m^2}{\mu_{mn}(\mu_{mn}{}^2 - m^2)} \tanh\left(\frac{\mu_{mn} H}{a}\right) + \frac{1 - (H/a)}{\cosh^2(\mu_{mn} H/a)} \right]$$

$$\operatorname{Re} \lambda_{mn} = -\frac{\nu^{1/2} g A_{mn}}{2\sqrt{2}\, \omega_{mn}^{3/2}}, \qquad \omega_{mn}{}^2 = \frac{\mu_{mn} g}{a} \tanh\left(\frac{\mu_{mn} H}{a}\right) \qquad \begin{pmatrix} m = 0, 1, \ldots \\ n = 1, 2, \ldots \end{pmatrix} \tag{3.11}$$

In the particular case $m = 1$ and $H \gg a$ an analogous equation for the coefficient of damping $\operatorname{Re} \lambda_{1n}$ was obtained in an approximate manner by a different method in [8].

In [9] experimental relationships are presented for the damping coefficient of characteristic oscillations of a fluid in a cylinder as functions of H/a and the Reynolds' number for the principal oscillation ($m = 1$, $n = 1$, $\mu_{11} = 1.841$). Results of calculations by means of Equations (3.11) are, qualitatively, in complete agreement with experimental data for various H/a and ν. The theoretical values of damping coefficients $\operatorname{Re} \lambda_{11}$, computed from Equation (3.11) are approximately equal to experimental values obtained in [9] multiplied by 0.71. This ratio is maintained for various ν and H/a. The same discrepancy between theoretical and experimental data for $H = \infty$ is noted in [9].

As an example of a vessel with varying depth we shall examine a channel with flat, mutually perpendicular walls inclined at an angle of 45° to the vertical. The region D occupied by the fluid is defined by inequalities $|y| - a \leqslant z \leqslant 0$, where $a > 0$ is the maximum depth of channel equal to one half its width. The characteristic oscillation of an ideal fluid in such a channel, corresponding to the smallest frequency (principal form of oscillation), is described by the potential

$$\Phi_1 = (y + a)\, z, \qquad \omega_1{}^2 = g / a \tag{3.12}$$

Relationships (3.12) are presented in [1] and it is not difficult to verify directly that they satisfy Equation (2.5). Substituting (3.12) into (2.21) and taking into account the remark at the end of section 2 (here S is a broken line $z = |y| - a$ for $|y| \leqslant a$ and Σ is a section $|y| \leqslant a$ of the y-axis) we obtain after elementary calculations

$$\lambda_1 = \pm i \sqrt{g/a} - (1 \pm i)(g\nu^2/a^5)^{1/4}$$

4. Let us examine the behavior of the solution of problem (1.6) near the contour Γ, the line of contact of free surface Σ and walls S. Let us select the origin of coordinates O at some point of contour Γ, orient the z-axis vertically upward, the x-axis along the tangent to Γ and the y-axis along the inward normal to Γ in the plane Σ (fig. 1). Limiting ourselves to a small region around O, we replace the surface S by the plane tangent to S at the point O. The curve Γ is replaced by its tangent at O, i.e. by the x-axis, and we shall examine the plane motion in the vertical yz-plane.

Equations and boundary conditions (1.6) take the form

$$\lambda \mathbf{u} = -\nabla q + \nu \Delta \mathbf{u}, \qquad \operatorname{div} \mathbf{u} = 0 \text{ in } D, \qquad \mathbf{u} = 0 \text{ when } z = -y \tan \delta$$
$$u_z = \frac{\lambda}{g} q - \frac{2\lambda\nu}{g} \frac{\partial u_z}{\partial z}, \qquad \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0, \qquad f = \frac{u_z}{\lambda} \text{ when } z = 0 \tag{4.1}$$

where $\mathbf{u}$ is a two-dimensional vector with components u_y and u_z and δ is the angle between the surface of walls S and the horizontal plane at the point O. On the basis of continuity equation it is possible to introduce the stream function Ψ such that $u_y = \partial\Psi / \partial z$, and $u_z = -\partial\Psi / \partial y$. Changing to stream function Ψ and introducing polar coordinates r and θ in the yz-plane, we rewrite relationships (4.1) in the form

$$q_r = \frac{1}{r}\left[\frac{\nu}{r}(r\Psi_r)_r + \frac{\nu}{r^2}\Psi_{\theta\theta} - \lambda\Psi\right]_\theta, \qquad \frac{q_\theta}{r} = -\left[\frac{\nu}{r}(r\Psi_r)_r + \frac{\nu}{r^2}\Psi_{\theta\theta} - \lambda\Psi\right]_r$$
$$\text{when } -\delta \leqslant \theta \leqslant 0$$
$$g\Psi_r + \lambda q + 2\lambda\nu(\Psi_\theta / r)_r = 0, \qquad r\Psi_r + \Psi_{\theta\theta} = r^2\Psi_{rr} \tag{4.2}$$
$$f = -\Psi_r / \lambda \text{ when } \theta = 0, \qquad \Psi_r = \Psi_\theta = 0 \text{ when } \theta = -\delta \qquad (y = r\cos\theta,\ z = r\sin\theta)$$

Here subscripts r and θ designate partial derivatives. Function $f(r)$, as before, determines the rise of free surface. We are looking for a solution of problem (4.2) in the form

$$\Psi(r, \theta) = r^{1+k} M(\theta) + \ldots \tag{4.3}$$

as $r \to 0$, where dots indicate terms of higher order with respect to r. In this connection velocity components are of order r^k when $r \to 0$, components of stress tensor have the order r^{k-1} and the force acting on the wall has the order r^k. Requiring boundedness of velocities we will assume $k > 0$. Substituting (4.3) into the first equation of (4.2) we shall determine the function q in the form

$$q = q_0(\theta) + [\nu r^{k-2} / (k-1)][M'' + (k+1)^2 M]' + \ldots \quad (k \neq 1) \tag{4.4}$$

where primes indicate derivatives with respect to θ. We substitute Expressions (4.3) and (4.4) into the second equation of (4.2)

$$q_0' + [\nu r^{k-1} / (k-1)]\,[M'' + (k+1)^2 M]'' + \nu(k-1)\, r^{k-1}[M'' +$$
$$+ (k+1)^2 M] + \ldots = 0$$

From here we obtain for $k \neq 1$ an equation for the function M which is easy to solve

$$[M'' + (k+1)^2 M]'' + (k-1)^2\,[M'' + (k+1)^2 M] = 0 \tag{4.5}$$

$$M(\theta) = C_1 \sin(k+1)\theta + C_2 \cos(k+1)\theta + C_3 \sin(k-1)\theta + \\ + C_4 \cos(k-1)\theta$$

Here C_1, C_2, C_3, and C_4 are constants. Now we substitute (4.3) and (4.4) into boundary conditions (4.2) and equate coefficients of the principal powers of r. For $k \neq 1$ we obtain the conditions

$$M''' + (k+1)^2 M' + 2k(k-1)M' = 0, \quad M'' - (k^2-1)M = 0 \text{ when } \theta = 0 \\ M = 0, \quad M' = 0 \quad \text{when } \theta = -\delta \tag{4.6}$$

Substituting the general solution (4.5) for $M(\theta)$ into conditions (4.6) we obtain a system of linear homogeneous equations for constants C_i

$$C_1 + C_3 = 0, \quad C_2(k+1) + C_4(k-1) = 0$$
$$-C_1 \sin(k+1)\delta + C_2\cos(k+1)\delta - C_3 \sin(k-1)\delta + \\ + C_4\cos(k-1)\delta = 0 \tag{4.7}$$
$$C_1(k+1)\cos(k+1)\delta + C_2(k+1)\sin(k+1)\delta + \\ + C_3(k-1)\cos(k-1)\delta + C_4(k-1)\sin(k-1)\delta = 0$$

For a nontrivial solution to exist, it is necessary to set the determinant of system (4.7) equal to zero. A characteristic equation is obtained for the index k

$$(k^2+1)\cos(k+1)\delta\cos(k-1)\delta + \\ + (k^2-1)[\sin(k+1)\delta\sin(k-1)\delta - 1] = 0$$

which for $k > 0$, is reduced, by simple transformations, to the form

$$\cos k\delta = k \sin\delta \tag{4.8}$$

The smallest positive root k of Equation (4.8) is of interest because it determines the principal term of asymptotic expansion of the form (4.3). It is easy to convince oneself that such a root $k(\delta)$ exists for any angle δ in the interval $(0, \pi)$. The function $k(\delta)$ decreases monotonely from ∞ to 0.5 on variation of δ from 0 to π. Values of function $k(\delta)$ were determined by numerical solution of Equation (4.8) on an electronic computer. Some of the values found are presented in the table. A graph of function $k(\delta)$ is represented in fig. 2

$\delta/\pi =$	0	$1/6$	$1/4$	$1/3$	$1/2$	$2/3$	$3/4$	$5/6$	1
$k =$	∞	1.4504	1	0.7855	0.5946	0.5248	0.5101	0.5029	0.5

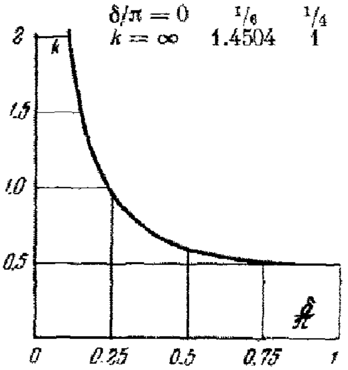

FIG. 2

We note that in the case $k = 1$ a logarithmic term appears in Equation (4.4); however the final result, Equation (4.8) does not change in this case.

If $k(\delta)$ satisfies Equation (4.8), then from system (4.7) it is possible to find constants C_i with accuracy to an arbitrary factor C.

$$C_1 = -C_3 = C(\sin k\delta + \cos\delta), \quad C_2 = C(1-k)\sin\delta \\ C_4 = C(1+k)\sin\delta$$

Since constants k and C_i are found, this at the same time completely determines the function $M(\theta)$ from (4.5) and the principal term of the solution of stream function (4.3). The constructed solution depends only on the angle of inclination of walls and, apparently, describes the character of singularity of solution of problem (1.6) near the contour Γ; subsequent members of expansion (4.3) must also depend on other geometrical properties of surface S (in particular its curvature). By means of (4.3) it is not difficult to determine asymptotic expressions when $r \to 0$ for velocities of fluid, stresses and other hydrodynamic quantities. Thus by virtue of (4.2) the elevation of the free surface $f(r)$ is proportional to r^k as $r \to 0$.

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